ELEN E6761: Communication Networks

Homework 7: Solutions

---

1. (a) Since TCP acts in AIMD phase and there is no packet loss, in each RTT, window size increases by 1. So we need 6 RTTs to increase window size from 6 to 12, and the window size remains at 12 for one extra RTT.
   (b)Consider the time period starting from the beginning of the RTT that W=6 to the end of the RTT that W=12.

$$\begin{aligned} r_{avg} &= \frac{\text{total number of successfully transmitted packets}}{\text{time duration}} \\ &= \frac{6+7+...+12}{7 \times \text{RTT}} \\ &= \frac{\frac{18\times 7}{2}}{7 \times \text{RTT}} \\ &= 9 \text{ Pkts/RTT} \end{aligned}$$

2. (a)

$$\begin{aligned} q &= \frac{\text{number of packet losses in each period}}{\text{number of total packets transmitting in each period}} \\ &= \frac{1}{\frac{W}{2} + (\frac{W}{2}+1) + ... + W} \\ &= \frac{1}{\frac{3}{8}W^2 + \frac{3}{4}W} \end{aligned}$$

   (b)

$$\begin{aligned} r_{avg} &= \frac{\text{total number of successfully transmitted packets}}{\text{time duration}} \\ &= \frac{\frac{3}{8}W^2 + \frac{3}{4}W - 1}{(\frac{W}{2}+1) \times \text{RTT}} \end{aligned}$$

   Assuming W is very large, or equivalently q is very small, we have:

$$r_{avg} = \frac{\frac{3}{4}W}{\text{RTT}} = \frac{\sqrt{\frac{3}{2}}}{\text{RTT}\sqrt{q}} \simeq \frac{1.22}{\text{RTT}\sqrt{q}}$$

3.

$$\begin{aligned} \frac{dx_i}{dt} &= x_i(t-T_i)(1-q_i(t))\frac{1}{w_i(t)} - x_i(t-T_i)q_i(t)\frac{1}{w_i(t)} \\ &= \frac{x_i(t-T_i)}{T_i x_i(t)}(1-2q_i(t)) \end{aligned} \quad (1)$$

Note that $T_1 = T_2 = T$ and $q_1(t) = q_2(t) = q(t)$. Suppose the equilibrium exists, it means that

$$x_i(t) \to x_i \text{ , as } t \to \infty$$
$$q(t) \to q \text{ , as } t \to \infty$$

then:

$$\frac{(1-2q)}{T} = 0 \Rightarrow q = \frac{1}{2}$$

Now consider a $M/M/1/B$ queue as our model for single link. Then we have that:

$$q = \frac{(1-\rho)\rho^B}{1-\rho^{B+1}} = \frac{1}{2}$$

Now, we can show that when buffer size goes to infinity, ρ goes to 1. The proof is same as what we have in AIMD TCP in lecture note. So AIAD is efficient.

Go back to (1), assume that t is large and $x_i(t - T_i) \simeq x_i(t)$. Since RTT and $q_i(t)$ are the same for both users,

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{1-2q(t)}{T} \Rightarrow x_1(t) = \alpha(t) + x_1(0) \\ \frac{dx_1}{dt} &= \frac{1-2q(t)}{T} \Rightarrow x_2(t) = \alpha(t) + x_2(0) \end{aligned} \tag{2}$$

where $\alpha(t) = \frac{1}{T}\int_0^t (1-2q(s))ds$. As it is clear from (2), if the initial rates for these two users are not equal, the rate will be different in any time, and the AIAD TCP does not converge to equal rates. So it is not fair.

4. I) If $\{X_r\}$ is a max-min fair allocation, every user has at least one bottleneck link.
We show this by contradiction. We first claim that there is at least one link in any route which is used fully. (that means $\forall r$, $\exists l$ s.t. $y_l = c_l$). To show this suppose that there is a route that none of its links is full. Then we can increase the rate of user r until one of the link along the route r becomes full. This gives another allocation which does not make a "poor" user, "poorer", which is contradiction with assumption that says $\{X_r\}$ is max-min fair.
To continue, suppose route(user) j does not have bottleneck link. Hence based on the above claim, there are a set of links S_j in the route of user j which are full but are not the bottleneck of user j, i.e., every link $l \in S_j$ is a bottleneck for another user $u(l) \neq j$ which shares the link l, and $x_{u(l)} > x_j$. Now, we make another allocation, $\{y_r\}$, by decreasing $x_{u(l)}$, $l \in S_j$, by ϵ and increasing x_j by ϵ, for some ϵ very small. Hence $\{y_r\}$ is still feasible and we have made user j richer, without making "poorer" users "poorer", which is contradiction with the assumption that $\{X_r\}$ is max-min fair.
II) If $\{X_r\}$ is an allocation such that every user has at least one bottleneck link, then $\{X_r\}$ is a max-min fair allocation.
To show this, take any other allocation, $\{y_r\}$ such that $\exists k$; $y_k > x_k$. Since user k has at least one bottleneck link l; either we should decrease other users' rates which are "poorer"

than user k to achieve $\{y_r\}$, if user k is sharing its bottleneck link with these users, or if k is the only user of this link l, we cannot increase its rate since it is already full. Hence $\{X_r\}$ is max-min fair.

5. a) Proportional Fair:

$$\begin{aligned}\max_{x_1,x_2,x_3} \quad & \log x_1 + \log x_2 + \log x_3 \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & x_1 + x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0\end{aligned}$$

Maximization of concave function under convex constraints is a convex optimization problem. Also, Slater's condition is satisfied, so strong duality holds. Suppose $x_1, x_2, x_3 \geq 0$ will be satisfied:

$$L(X, \lambda) = \log x_1 + \log x_2 + \log x_3 - \lambda_1(x_1 + x_2 - 2) - \lambda_2(x_1 + x_3 - 1)$$

KKT Conditions:

$$\begin{gathered} x_1^* + x_2^* \leq 2,\ x_1^* + x_3^* \leq 1 \\ \lambda_1^* \geq 0,\ \lambda_2^* \geq 0 \\ \lambda_1^*(x_1^* + x_2^* - 2) = 0 \\ \lambda_2^*(x_1^* + x_3^* - 1) = 0 \\ \left.\frac{\partial L}{\partial x_i}\right|_{x_i = x_i^*} = 0 \Rightarrow \frac{1}{x_1^*} - \lambda_1^* - \lambda_2^* = 0 \\ \frac{1}{x_2^*} - \lambda_1^* = 0 \\ \frac{1}{x_3^*} - \lambda_2^* = 0 \end{gathered}$$

Above equations yield:

$$\lambda_1^* = \frac{3 - \sqrt{3}}{2},\ \lambda_2^* = \sqrt{3}$$
$$x_1^* = 1 - \frac{\sqrt{3}}{3},\ x_2^* = 1 + \frac{\sqrt{3}}{3},\ x_3^* = \frac{\sqrt{3}}{3}$$

Minimum Delay:

$$\begin{aligned}\min_{x_1,x_2,x_3} \quad & \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & x_1 + x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0\end{aligned}$$

Minimization of convex optimization under convex constraints is a convex optimization problem. Also, Slater's condition is satisfied, so strong duality holds. Suppose $x_1, x_2, x_3 \geq 0$ will be satisfied:

$$L(X, \lambda) = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \lambda_1(x_1 + x_2 - 2) + \lambda_2(x_1 + x_3 - 1)$$

KKT Conditions:

$$\begin{aligned}
x_1^* + x_2^* \leq 2,\ \ x_1^* + x_3^* \leq 1 \\
\lambda_1^* \geq 0,\ \ \lambda_2^* \geq 0 \\
\lambda_1^*(x_1^* + x_2^* - 2) = 0 \\
\lambda_2^*(x_1^* + x_3^* - 1) = 0 \\
\left.\frac{\partial L}{\partial x_i}\right|_{x_i = x_i^*} = 0 \Rightarrow \frac{-1}{x_1^{*2}} + \lambda_1^* + \lambda_2^* = 0 \\
\frac{-1}{x_2^{*2}} + \lambda_1^* = 0 \\
\frac{-1}{x_3^{*2}} + \lambda_2^* = 0
\end{aligned}$$

Above equations yield:

$$\lambda_1^* \simeq 0.44,\ \ \lambda_2^* \simeq 3.79$$

$$x_1^* \simeq 0.49,\ \ x_2^* \simeq 1.51,\ \ x_3^* \simeq 0.51$$

Max-min Fair: Every user has a bottleneck link, user 2's bottleneck link is link A and user 3's bottleneck link is link B. If user 1's bottleneck link is link B, then $x_1^* = 0.5,\ x_2^* = 1.5,\ x_3^* = 0.5$. This allocation is max-min fair based on result of problem 4.

b) Updates for proportional fair:

$$\begin{aligned}
x_1^{(k)} &= \frac{1}{\lambda_1^{(k)} + \lambda_2^{(k)}} \\
x_2^{(k)} &= \frac{1}{\lambda_1^{(k)}} \\
x_3^{(k)} &= \frac{1}{\lambda_2^{(k)}} \\
\lambda_1^{(k+1)} &= \left(\lambda_1^{(k)} + \delta^{(k)}(x_1^{(k)} + x_2^{(k)} - 2)\right)^+ \\
\lambda_2^{(k+1)} &= \left(\lambda_2^{(k)} + \delta^{(k)}(x_1^{(k)} + x_3^{(k)} - 1)\right)^+
\end{aligned}$$