ELEN E6761: Communication Networks

Homework 4: Solutions

---

1. $\{(d_1(t), d_2(t)); t = 1, 2, ...\}$ is a Markov chain with state space $\{(1,1); (1,2); (2,1); (2,2)\}$

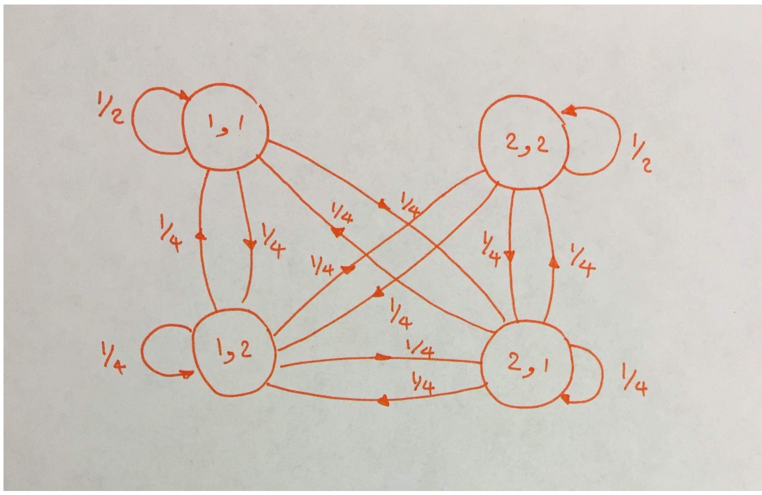

Figure 1: The Markov Chain

$$\mathbb{P} = \begin{array}{c} \\ 11 \\ 12 \\ 21 \\ 22 \end{array} \begin{array}{c} \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \\ \begin{bmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/2 \end{bmatrix} \end{array}$$

The Markov chain is irreducible, aperiodic and positive recurrent. So the equilibrium probability vector exists which is unique and satisfies Global Balance Equations: $\pi = \pi\mathbb{P}$.

$$(\frac{1}{4} + \frac{1}{4})\pi_{11} = \frac{1}{4}\pi_{12} + \frac{1}{4}\pi_{21}$$
$$(\frac{1}{4} + \frac{1}{4})\pi_{22} = \frac{1}{4}\pi_{12} + \frac{1}{4}\pi_{21}$$
$$(\frac{1}{4} + \frac{1}{4} + \frac{1}{4})\pi_{12} = \frac{1}{4}\pi_{11} + \frac{1}{4}\pi_{22} + \frac{1}{4}\pi_{21}$$
$$(\frac{1}{4} + \frac{1}{4} + \frac{1}{4})\pi_{21} = \frac{1}{4}\pi_{11} + \frac{1}{4}\pi_{22} + \frac{1}{4}\pi_{12}$$

By symmetry $\pi_{11} = \pi_{22}$ and $\pi_{12} = \pi_{21}$. Then the equations are easily solved, and we get:

$$\pi_{11} = \pi_{12} = \pi_{21} = \pi_{22} = \frac{1}{4}$$

(b)

$$\begin{aligned} r &= \sum_{i,j} \pi_{ij} r_{ij} \\ &= \frac{1}{4}(2+2+1+1) \\ &= \frac{3}{2} \end{aligned}$$

So the loss due to HOL blocking is $2 - 3/2 = 1/2$ pkt/time slot.

2. Consider VOQ_{11}. At the beginning of each time slot, if both VOQ_{12} and VOQ_{21} are empty, then the priority algorithm transfers a packet from VOQ_{11}. Since $Q_{12}(0) = Q_{21}(0) = 0$, the probability that both VOQ_{12} and VOQ_{21} are empty at any time slot is equal to $(1-\lambda_{12})(1-\lambda_{21})$. The same argument also holds for VOQ_{22}. So

$$\begin{aligned} \text{stability condition for VOQ}_{11} \text{ and VOQ}_{22}\text{: } & \lambda_{11} < (1-\lambda_{12})(1-\lambda_{21}) \\ & \lambda_{22} < (1-\lambda_{12})(1-\lambda_{21}) \end{aligned}$$

Note that there are two other necessary conditions on input and output ports based on physical constraint on packet transmission in each time slot, which are loser than the above stability conditions for VOQs.

For MaxWeight algorithm (which we know it is optimal), we know that the stability condition is the following

$$\begin{aligned} & \lambda_{11} + \lambda_{12} < 1 \Rightarrow \lambda_{11} < 1 - \lambda_{12} \\ & \lambda_{11} + \lambda_{21} < 1 \Rightarrow \lambda_{11} < 1 - \lambda_{21} \\ & \Rightarrow \lambda_{11} < \min(1-\lambda_{12}, 1-\lambda_{21}) \end{aligned}$$

and a similar argument for VOQ_{22}

$\Lambda_{\text{MaxWeight}} = \{(\lambda_{11}, \lambda_{22}); \lambda_{11} \geq 0, \lambda_{22} \geq 0, \lambda_{11} < \min(1-\lambda_{12}, 1-\lambda_{21}), \lambda_{22} < \min(1-\lambda_{12}, 1-\lambda_{21})\}$

Clearly priority scheduling algorithm is not throughput optimal, since there are some $(\lambda_{11}, \lambda_{22})$ which are in $\Lambda_{\text{MaxWeight}}$ but are not in $\Lambda_{\text{priority policy}}$.

3.

$$\begin{aligned} V(Q(t+1)) - V(Q(t)) &= \sum_{ij} Q_{ij}^3(t+1) - \sum Q_{ij}^3(t) \\ &= \sum (Q_{ij}(t) + a_{ij}(t) - M_{ij}^*)^{+^3} - \sum Q_{ij}^3(t) \\ &\overset{*}{=} \sum \Big(Q_{ij}^3(t) + 3Q_{ij}^2(t)(a_{ij} - M_{ij}^*) + 3Q_{ij}(t)(a_{ij} - M_{ij}^*)^2 \\ &\quad + (a_{ij} - M_{ij}^*)^3 \Big) - \sum Q_{ij}^3(t) \end{aligned}$$

*: If $Q_{ij}(t) \geq 1$ (to be able to omit "+" symbol in $(Q_{ij}(t) + a_{ij}(t) - M_{ij}^*)^+$).

$$\begin{aligned} D &= \mathbb{E}\left[V(Q(t+1)) - V(Q(t)) | Q(t) = Q\right] \\ &= \sum 3Q_{ij}^2 \mathbb{E}\left[(a_{ij} - M_{ij}^*)\right] + 3Q_{ij}\mathbb{E}\left[(a_{ij} - M_{ij}^*)^2\right] + \mathbb{E}\left[(a_{ij} - M_{ij}^*)^3\right] \\ &\leq \sum 3(\lambda_{ij} - M_{ij}^*)Q_{ij}^2 + 3(\lambda_{ij} + 1)Q_{ij} + \lambda_{ij} \\ &\leq \sum 3((1+\epsilon)\lambda_{ij} - M_{ij}^*)Q_{ij}^2 - 3\epsilon\lambda_{ij}Q_{ij}^2 + 3(\lambda_{ij}+1)Q_{ij} + C \end{aligned}$$

Since for every matrix λ there is a matrix μ in convex hull of matching matrices such that $\lambda < \mu$, we can find ϵ such that $(1+\epsilon)\lambda \leq \mu$. Since for every matching matrix M we have $\sum M^*_{ij} Q^2_{ij} \geq \sum M_{ij} Q^2_{ij}$, and μ is a convex combination of matching matrices, then $\sum M^*_{ij} Q^2_{ij} \geq \sum \mu_{ij} Q^2_{ij}$. So $\sum((1+\epsilon)\lambda_{ij} - M^*_{ij})Q^2_{ij}$ is negative.
Then:

$$D \leq \sum_{ij} -3\epsilon\lambda_{ij} Q^2_{ij} + 3(\lambda_{ij}+1)Q_{ij} + C$$

Note that coefficient of Q^2_{ij} is negative. So by taking Q_{ij} large enough, we can make each term of this summation negative. Then we can conclude that for such Q_{ij}'s (which are outside a finite set), D is negative.
The formal argument is as follows. Suppose

$$Q_{ij} \geq \frac{3(\lambda_{ij}+1)}{\epsilon\lambda_{ij}}, \tag{1}$$

then

$$-3\epsilon\lambda_{ij} Q^2_{ij} + 3(\lambda_{ij}+1)Q_{ij} \leq -2\epsilon\lambda_{ij} Q^2_{ij},$$

and consequently,

$$D \leq -2\epsilon \sum_{ij} \lambda_{ij} Q^2_{ij} + C$$

Then for any $\delta > 0$, if we further have

$$-2\epsilon \sum \lambda_{ij} Q^2_{ij} + C \leq -\delta, \tag{2}$$

then the drift $D \leq -\delta$. Thus, we need both (1) and (2) to hold simultaneously, as depicted below (in two dimensions for simplicity)