ELEN E6761: Communication Networks

Homework 4: Solutions

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1. Slotted ALOHA with variable packet lengths:
   X_i's are iid, and $\mathbb{E}[X_i] = \bar{X} = 1$
   (a)

$$Y = \max(X_1, X_2) \le X_1 + X_2 \to \bar{Y} \le \bar{X} + \bar{X} = 2\bar{X}$$

(b)

$$\mathbb{E}[\text{cycle length}|\text{collision of two packets}] = \mathbb{E}[Y + \beta] = \mathbb{E}[Y] + \beta \le 2 + \beta$$

(c)
N: Number of attempted transmissions in this cycle.
N_1: Number of attempted transmissions from $N_k = n$ backlogged packets $\sim$ Binomial(n, q).
N_2: Number of packets that arrived in idol slot (with length β)$\sim$ Poisson$(\lambda\beta)$. These packets will be transmitted with probability 1.

$$N = N_1 + N_2 \Rightarrow \mathbb{E}[N] = \mathbb{E}[N_1] + \mathbb{E}[N_2] = nq + \lambda\beta$$

(d)

$$\begin{aligned}
\mathbb{E}[\text{cycle length}] = &\sum_{j=0}^{\infty} \mathbb{E}[\text{cycle length}|\text{number of attempted transmission} = j] \\
&\times \mathbb{P}(\text{number of attempted transmission} = j) \\
\simeq &\sum_{j=0}^{2} \mathbb{E}[\text{cycle length}|\text{number of attempted transmission} = j] \\
&\times \mathbb{P}(\text{number of attempted transmission} = j) \\
= &\beta e^{-g(n)} + \mathbb{E}[X + \beta] e^{-g(n)} g(n) + \mathbb{E}[Y + \beta] e^{-g(n)} \frac{g^2(n)}{2!} \\
\le &\beta e^{-g(n)} + (1 + \beta) g(n) e^{-g(n)} + (1 + \frac{\beta}{2}) g^2(n) e^{-g(n)}
\end{aligned}$$

(e)

$$\begin{aligned}
D_k &= \mathbb{E}[N_{k+1} - N_k | N_k = n] \\
&= \lambda \mathbb{E}[\text{cycle length}] - \mathbb{P}(\text{success}) \\
&\le \lambda \Big(\beta e^{-g(n)} + (1 + \beta) g(n) e^{-g(n)} + (1 + \frac{\beta}{2}) g^2(n) e^{-g(n)} \Big) - g(n) e^{-g(n)}
\end{aligned}$$

To ensure the drift is negative:

$$\lambda < \frac{g(n)}{\beta + (1 + \beta) g(n) + (1 + \frac{\beta}{2}) g^2(n)}$$

(f) To maximize the throughput (the largest arrival rate possible), we should maximize the right-hand-side of the above inequality.

Let $x = g(n)$ and $f(x) = \frac{x}{\beta+(1+\beta)x+(1+\frac{\beta}{2})x^2}$, then solving $\frac{df(x)}{dx}\Big|_{x=x^*} = 0$ to find the optimal x^* yields,

$$\frac{\beta + (1+\beta)x^* + (1+\frac{\beta}{2})x^{*^2} - \big(1+\beta) + 2(1+\frac{\beta}{2}x^*\big)x^*}{(\beta + (1+\beta)x^* + (1+\frac{\beta}{2})x^{*^2})^2} = 0$$

Hence,

$$\Rightarrow x^* = \pm\sqrt{\frac{\beta}{1+\frac{\beta}{2}}}$$

$$g(n) \geq 0 \Rightarrow g^*(n) = \sqrt{\frac{\beta}{1+\frac{\beta}{2}}} \simeq \sqrt{\beta(1-\frac{\beta}{2})} \simeq \sqrt{\beta}$$

(g)

$$g^*(n) = \sqrt{\beta} = \lambda\beta + nq^* \Rightarrow q^* \simeq \frac{\sqrt{\beta}}{n}$$

Putting everything together,

$$\lambda < \frac{\sqrt{\beta}}{\beta + (1+\beta)\sqrt{\beta} + (1+\frac{\beta}{2})\beta} \simeq \frac{1}{1+2\sqrt{\beta}}$$