ELEN E6761: Communication Networks

## Homework 3: Solutions

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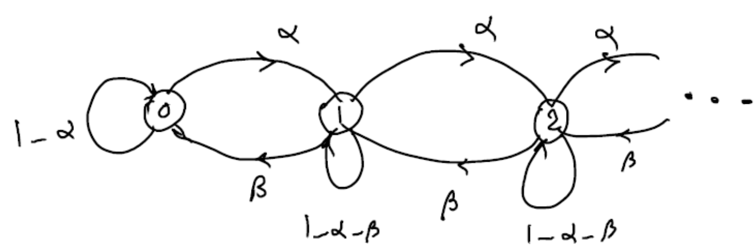
1. (a)

$$\mathbb{P}\left(\text{Success under this model}\right) = nq(1-q)^{n-1} \overset{*}{=} (1-\frac{1}{n})^{n-1} > \frac{1}{e}$$
$$\mathbb{P}\left(\text{Success in Geo/Geo/1}\right) = \frac{1}{e}$$

*: Since we have centralized control, $q = \frac{1}{n}$.
Hence, the mean time duration between successful packet transmissions is less than the time required to serve a packet in Geo/Geo/1 queue. Moreover, the order at which packets are served does not change the average delay. Hence, we can bound the average delay of ALOHA in this model by the average delay of Geo/Geo/1.

(b) Delay of the corresponding Geo/Geo/1 queue:
(i)



Global Balance Equations:

$$\alpha\pi_0 = \beta\pi_1$$
$$\alpha\pi_1 = \beta\pi_2$$
$$\dots$$
$$\sum_{i=0}^{\infty} \pi_i = 1$$

where $\alpha = \lambda(1-\frac{1}{e})$ and $\beta = \frac{1}{e}(1-\lambda)$. Note that $\pi$ exists if and only if $\alpha < \beta$, in this case

$$\pi_0 = 1 - \frac{\alpha}{\beta}$$
$$\pi_i = (\frac{\alpha}{\beta})^i \pi_0$$

Now we can calculate $\mathbb{E}\left[N\right]$:

$$\mathbb{E}\left[N\right] = \sum_{i=0}^{\infty} i\pi_i = \pi_0 \frac{\frac{\alpha}{\beta}}{(1-\frac{\alpha}{\beta})^2} = \frac{\frac{\alpha}{\beta}}{1-\frac{\alpha}{\beta}}$$

Note that we are looking at the system at the beginning of time slots, when there are no packets in transmission, i.e., all packets are *waiting* for transmissions. Hence, if we apply Little's Law:

$$\mathbb{E}[N] = \lambda \mathbb{E}[W] \Rightarrow \mathbb{E}[W] = \frac{\alpha}{\lambda(\beta - \alpha)} = \frac{e-1}{1-\lambda e}$$

(ii)

Note that since arrivals happen at the beginning of time slots, any packet in service has been already departed by the end of previous time slot so residual time $R = 0$. Now suppose there are $n$ packets already in system when the current packet arrives, then the waiting time of the newly arrived packet to start service (excluding its transmission time which is 1 time slots) is given by

$$\begin{aligned} W &= \sum_{i=1}^{n} x_i + x_{n+1} - 1 \\ \mathbb{E}[W] &= \mathbb{E}[N]\,\mathbb{E}[x] + \mathbb{E}[x] - 1 \\ &\overset{*}{=} \lambda \mathbb{E}[W]\, e + e - 1 \end{aligned}$$

Therefore

$$\mathbb{E}[W] = \frac{e-1}{1-\lambda e}$$

which is the same expression as in method (i). The expected delay is $\mathbb{E}[W] + 1$.

(c) Stability condition of the Geo/Geo/1 queue:

$$\alpha < \beta \Rightarrow \lambda < \frac{1}{e}$$

(d) Stability condition of the ALOHA in this model:

$$\text{Drift} = \mathbb{E}[N(t+1) - N(t) | N(t) = n] = \lambda - nq(1-q)^{n-1} \overset{*}{=} \lambda - (1 - \frac{1}{n})^{n-1} < \lambda - \frac{1}{e}$$

*: since we have centralized control for this model, so $q = \frac{1}{n}$.

To have a stable system, Drift should be negative for $n$ large enough. If $\lambda < \frac{1}{e}$ then obviously Drift is negative for all $n$ so $\lambda < \frac{1}{e}$ is a sufficient condition for stability. On the other hand, if we take $\lambda = \frac{1}{e} + \epsilon$, $\epsilon \geq 0$, then we can always find $n_0 = n_0(\epsilon)$ such that for all $n > n_0$ Drift is nonnegative. So $\lambda < \frac{1}{e}$ is the necessary and sufficient condition for stability.