ELEN E6761: Communication Networks

Homework 2: Solutions

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1. (a)
Global Balance Equations:

$$\begin{aligned}
&\pi_1 p_1 + \pi_1(1-p_1) = \pi_2(1-p_2) + \pi_3 p_3 \\
&\pi_2 p_2 + \pi_2(1-p_2) = \pi_3(1-p_3) + \pi_1 p_1 \\
&\pi_3 p_3 + \pi_3(1-p_3) = \pi_1(1-p_1) + \pi_2 p_2 \\
&\pi_1 + \pi_2 + \pi_3 = 1
\end{aligned}$$

Then:

$$\begin{aligned}
\pi_1 &= \frac{1-p_2(1-p_3)}{(2-p_1)(1-p_2(1-p_3)) + (1+p_2)(1-p_3(1-p_1))} \\
&= \frac{1-p_2+p_2p_3}{3-p_1-p_2-p_3+p_1p_2+p_1p_3+p_2p_3}
\end{aligned}$$

π_2 and π_3 can be written similarly.

(b)
$\{A\} = \{$a clockwise move that is then followed by one consecutive counterclockwise move$\}$

$$\mathbb{P}(A) = \sum_{i=1}^{3} \mathbb{P}(A|S_i)\,\pi_i = p_1(1-p_2)\pi_1 + p_2(1-p_3)\pi_2 + p_3(1-p_1)\pi_3$$

2. (a)

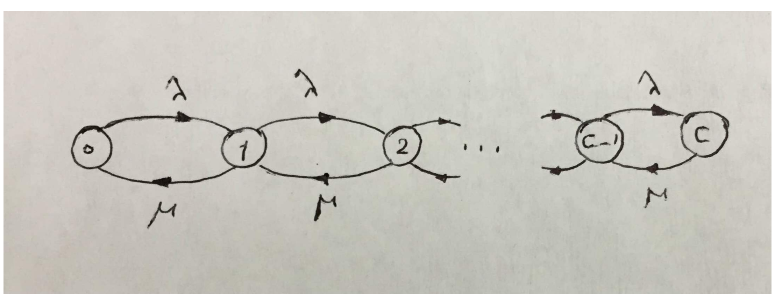

Figure 1: The Markov Chain of M/M/1/C Queue

Detailed Balance Equations:

$$\begin{aligned}
&\pi_0\lambda = \pi_1\mu \\
&\pi_1\lambda = \pi_2\mu \\
&... \\
&\pi_{c-1}\lambda = \pi_c\mu \\
&\sum_{i=0}^{c}\pi_i = 1
\end{aligned}$$

So

$$\pi_i = (\frac{\lambda}{\mu})^i \pi_0$$
$$\sum_{i=0}^{c} (\frac{\lambda}{\mu})^i \pi_0 = 1$$

Case 1: $\lambda = \mu$

$$\sum_{i=0}^{c} (\frac{\lambda}{\mu})^i \pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{c+1}$$
$$\pi_i = \pi_0 \text{ , for i=0 to c}$$

Case 2: $\lambda \neq \mu$
$\rho = \frac{\lambda}{\mu}$:

$$\sum_{i=0}^{c} (\frac{\lambda}{\mu})^i \pi_0 = 1 \Rightarrow \pi_0 = \frac{1-\rho}{1-\rho^{c+1}}$$
$$\pi_i = \rho^i \frac{1-\rho}{1-\rho^{c+1}} \text{ , for i=0 to c}$$

This system is stable for any ρ.
(b)
N: Number of customers in the system.

$$\mathbb{E}[N] = \sum_{i=0}^{c} i\pi_i$$

Case 1: $\lambda = \mu$

$$\mathbb{E}[N] = \frac{1}{c+1} \times \frac{c(c+1)}{2} = \frac{c}{2}$$

Case 2: $\lambda \neq \mu$
$\rho = \frac{\lambda}{\mu}$:

$$\begin{aligned}
\mathbb{E}[N] &= \sum_{i=0}^{c} i\rho^i \frac{1-\rho}{1-\rho^{c+1}} \\
&\overset{*}{=} \frac{\rho(1-\rho)}{1-\rho^{c+1}} \times \frac{-(c+1)\rho^c(1-\rho) + 1 - \rho^{c+1}}{(1-\rho)^2} \\
&= \frac{\rho}{1-\rho} - \frac{(c+1)\rho^{c+1}}{1-\rho^{c+1}}
\end{aligned}$$

* To find out $\sum_{i=0}^{c} i\rho^i$:

$$\sum_{i=0}^{c} \rho^i = \frac{1-\rho^{c+1}}{1-\rho}$$

Then we take the derivative of both sides with respect to ρ:

$$\sum_{i=0}^{c} i\rho^{1-i} = \frac{-(c+1)\rho^c(1-\rho)+1-\rho^{c+1}}{(1-\rho)^2}$$

So $\sum_{i=0}^{c} i\rho^i = \rho \times \sum_{i=0}^{c} i\rho^{1-i} = \rho \times \frac{-(c+1)\rho^c(1-\rho)+1-\rho^{c+1}}{(1-\rho)^2}$.

(c)

When a customer sees the system in state c, he/she cannot enter the system. By PASTA, drop probability is equal to π_c. So the effective rate of arrival for this system is $\lambda(1-\pi_c)$. Then by Little's law:

$$\lambda_{\text{eff}} \times \mathbb{E}[D] = \mathbb{E}[N] \longrightarrow \mathbb{E}[D] = \frac{\mathbb{E}[N]}{\lambda(1-\pi_c)}$$

3. (a)

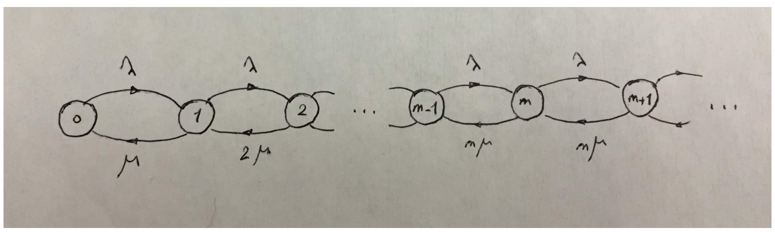

Figure 2: The Markov Chain of M/M/∞ Queue

Detailed Balance Equations:

$$\begin{aligned}
&\sum_{i=0}^{\infty} \pi_i = 1 \\
&\lambda\pi_0 = \mu\pi_1 \\
&\lambda\pi_1 = 2\mu\pi_2 \\
&\ldots \\
&\lambda\pi_{m-1} = m\mu\pi_m \\
&\lambda\pi_m = m\mu\pi_{m+1} \\
&\ldots
\end{aligned}$$

Let $\rho = \frac{\lambda}{\mu}$, then:

$$\pi_i = \frac{\rho^i}{i!}\pi_0 \text{ , for i=0 to m}$$

$$\pi_i = (\frac{\rho}{m})^{i-m}\pi_m = (\frac{\rho}{m})^{i-m}\frac{\rho^m}{m!}\pi_0 \text{ , for i=m+1, m+2, ...}$$

Then to find π_0:

$$\sum_{i=0}^{\infty} \pi_i = 1 \Rightarrow \sum_{i=0}^{m} \frac{\rho^i}{i!}\pi_0 + \sum_{i=m+1}^{\infty} (\frac{\rho}{m})^{i-m}\frac{\rho^m}{m!}\pi_0 = 1$$

$$\overset{*}{\Rightarrow} \pi_0 = \frac{1}{\sum_{i=0}^{m} \frac{\rho^i}{i!} + \frac{\rho^{m+1}}{m \times m!}\frac{1}{1-\frac{\rho}{m}}}$$

* For $\rho < m$, which is the stability condition.

(b)

$$\begin{aligned}\mathbb{P}\left(\text{a customer has to wait}\right) &= \mathbb{P}\left(\text{a customer sees m customer or more in the system}\right)\\ &\overset{*}{=} \mathbb{P}\left(\text{m persons or more in the system}\right)\\ &= \sum_{i=m}^{\infty}\pi_i = 1 - \sum_{i=0}^{m-1}\pi_i = 1 - \pi_0\sum_{i=0}^{m-1}\frac{\rho^i}{i!}\end{aligned}$$

* By using PASTA.

(c)

$$\pi_0 = \frac{1}{\sum_{i=0}\infty\frac{\rho^i}{i!}} = e^{-\rho} \longrightarrow \pi_i = e^{-\rho}\frac{\rho^i}{i!}$$

Note that this is a Possion distribution with mean ρ. Hence, the number of customers in the $M/M/\infty$ system in steady state is simply a Poisson random variable with mean $\rho = \frac{\lambda}{\mu}$.

Also, the system is stable for all $\rho < \infty$. And, $\mathbb{P}\left(\text{a customer has to wait}\right) \longrightarrow 0$

4. $N(t) \sim \text{Poisson}(\alpha)$, $\mathbb{P}\left(\text{transmission}\right) = \frac{1}{\alpha}$

(a)

$$\begin{aligned}\mathbb{P}\left(\text{slot is idle}\right) &= \mathbb{P}\left(\text{no packet is transmitted}\right)\\ &= \sum_{n=0}^{\infty}\mathbb{P}\left(\text{no packet is transmitted}|N(t)=n\right)\mathbb{P}\left(N(t)=n\right)\\ &= \sum_{n=0}^{\infty}(1-\frac{1}{\alpha})^n e^{-\alpha}\frac{\alpha^n}{n!} = e^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha-1)^n}{n!}\\ &= e^{-\alpha}e^{\alpha-1} = \frac{1}{e}\end{aligned}$$

(b)

$$\begin{aligned}\mathbb{P}\left(N=n|\text{slot is idle}\right) &= \frac{\mathbb{P}\left(\text{slot is idle}|N=n\right)\mathbb{P}\left(N=n\right)}{\mathbb{P}\left(\text{slot is idle}\right)}\\ &= \frac{(1-\frac{1}{\alpha})^n e^{-\alpha}\frac{\alpha^n}{n!}}{\frac{1}{e}} = e^{-(\alpha-1)}\frac{(\alpha-1)^n}{n!}\\ &\sim \text{Poisson}(\alpha-1)\end{aligned}$$

(c)

$$\begin{aligned}\mathbb{P}\left(\text{slot is successful}\right) &= \mathbb{P}\left(\text{only one station transmits}\right)\\ &= \sum_{n=1}^{\infty}\mathbb{P}\left(\text{only one station transmits}|N(t)=n\right)\mathbb{P}\left(N(t)=n\right)\\ &= \sum_{n=1}^{\infty}\binom{n}{1}\frac{1}{\alpha}(1-\frac{1}{\alpha})^{n-1}e^{-\alpha}\frac{\alpha^n}{n!} = \sum_{n=1}^{\infty}e^{-\alpha}\frac{(\alpha-1)^{n-1}}{(n-1)!}\\ &= e^{-1}\sum_{n=0}^{\infty}e^{-(\alpha-1)}\frac{(\alpha-1)^n}{n!} = \frac{1}{e}\end{aligned}$$

(d)

$$
\begin{aligned}
\mathbb{P}\left(N = n+1 \middle| \text{slot is successful}\right) &= \frac{\mathbb{P}\left(\text{slot is successful} \middle| N = n+1\right)\mathbb{P}\left(N = n+1\right)}{\mathbb{P}\left(\text{slot is successful}\right)} \\
&= \frac{(n+1)\frac{1}{\alpha}(1-\frac{1}{\alpha})^n e^{-\alpha}\frac{\alpha^{n+1}}{(n+1)!}}{\frac{1}{e}} = e^{-(\alpha-1)}\frac{(\alpha-1)^n}{n!} \\
&\sim \text{Poisson}(\alpha-1)
\end{aligned}
$$