ELEN E6761: Communication Networks

Homework 1: Solutions

1.

$$
\begin{aligned}
\mathbb{P}(X \le Y) &= \int_0^\infty \mathbb{P}(X \le Y | X = x) f_X(x) dx \\
&= \int_0^\infty \mathbb{P}(x \le Y | X = x) f_X(x) dx \\
&= \int_0^\infty \mathbb{P}(x \le Y) f_X(x) dx \text{ (because X and Y are independent)} \\
&= \int_0^\infty e^{-\mu x} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda + \mu}
\end{aligned}
$$

2. (a) One of the first two customers departs first which is then replaced by the third customer. From that point onward, the service times of the remaining customers are still independent exponential random variables with the same parameter (by the memoryless property of exponential random variable). Hence either of them will depart next with equal probability.

   In mathematical language: let T_i be the service time of the i-th customer, $i = 1, 2, 3$.

$$
\begin{aligned}
\mathbb{P}(\text{the person is the last one to exit}) &= \mathbb{P}(T_3 > |T_1 - T_2|) \\
&= \frac{\lambda}{\lambda + \lambda} = 0.5
\end{aligned}
$$

   where the last equality follows from the fact that $|T_1 - T_2|$ is still exponential with the same parameter and using the result of problem 1.

   (b) $\frac{1}{\lambda} = 2$ minutes

$$
\mathbb{E}[T] = \mathbb{E}[\min(T_1, T_2)] + \mathbb{E}[T_3] \overset{*}{=} 1^{min} + 2^{min} = 3^{min}
$$

   *: According to question 2 of HW0, $\min(T_2, T_1) \sim \text{Exp}(2\lambda)$

3. $X \sim \text{Exp}(\mu), Y \sim \text{Exp}(\lambda)$, X and Y are independent. Let T be the random variable denoting the time between replacements.

   Method 1: Using conditional expectation

$$
\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[T|X \le Y]\,\mathbb{P}(X < Y) + \mathbb{E}[T|X > Y]\,\mathbb{P}(X > Y) \\
&= \mathbb{E}[Y]\,\mathbb{P}(X < Y) + (\mathbb{E}[T + Y])\mathbb{P}(X > Y) \\
&= \frac{1}{\lambda} \times \frac{\mu}{\mu + \lambda} + (\mathbb{E}[T] + \frac{1}{\lambda}) \times \frac{\lambda}{\mu + \lambda}
\end{aligned}
$$

   Therefore,

$$
\mathbb{E}[T] = \frac{1}{\mu} + \frac{1}{\lambda}
$$

Method 2: the number of checking points to get a replacement is geometric with parameter p, where

$$p = \mathbb{P}\left(\text{A checking point is a replacement point}\right) = \mathbb{P}\left(X < Y\right) = \frac{\mu}{\mu+\lambda}$$

$$\frac{1}{p} = \text{Average \# of checking points to get the first occurrence of replacement point.}$$

$$\mathbb{E}\left[T\right] = \frac{1}{p} \times \mathbb{E}\left[Y\right] = \frac{\mu+\lambda}{\mu} \times \frac{1}{\lambda} = \frac{1}{\mu} + \frac{1}{\lambda}$$

Method 3: You can also consider the replacement process to be the checking process sampled with probability $\frac{\mu}{\mu+\lambda}$. According to splitting property of Poisson process, replacement process is another Poisson process with rate $\frac{\mu}{\mu+\lambda} \times \lambda$. So the inter arrival times will be exponentially distributed with mean $\frac{1}{\mu} + \frac{1}{\lambda}$.

4. We show that $N(t)$ satisfies the first definition of Poisson process, with rate $\lambda_1 + \lambda_2$.

   i)

$$N(0) = N_1(0) + N_2(0) = 0$$

   ii) Let define $S = \min(X_1, X_2)$ where $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \exp(\lambda_2)$. Let $N(t) = N_1(t)+N_2(t) = k$ and $S = s$ then $N(t+s) = N_1(t+s)+N_2(t+s) = k+1$ because either N_1 or N_2 increases by 1 at time s. From question 2 of HW0, $S \sim \text{Exp}(\lambda_1 + \lambda_2)$. Since X_i's (i=1,2) are iid exponentially distributed, S's are iid exponentially distributed.

5. N(t) is a Poisson process with rate λ:

   (a) $\mathbb{P}\left(N(15) = 2 | N(60) = 2\right)$:

$$\begin{aligned}
\mathbb{P}\left(N(15) = 2 | N(60) = 2\right) &= \frac{\mathbb{P}\left(N(15) = 2, N(60) = 2\right)}{\mathbb{P}\left(N = 60\right)} = \frac{\mathbb{P}\left(N(15) = 2, N(60) - N(15) = 0\right)}{\mathbb{P}\left(N = 60\right)} \\
&= \frac{\mathbb{P}\left(N(15) = 2\right)\mathbb{P}\left(N(45) = 0\right)}{\mathbb{P}\left(N = 60\right)} \\
&= \frac{1}{16}
\end{aligned}$$

   (b) $\mathbb{P}\left(N(15) \geq 1 | N(60) = 2\right)$:

$$\begin{aligned}
\mathbb{P}\left(N(15) \geq 1 | N(60) = 2\right) = 1 - \mathbb{P}\left(N(15) = 0 | N(60) = 2\right) &= 1 - \frac{\mathbb{P}\left(N(15) = 0, N(60) = 2\right)}{\mathbb{P}\left(N = 60\right)} \\
&= 1 - \frac{\mathbb{P}\left(N(15) = 0, N(60) - N(15) = 2\right)}{\mathbb{P}\left(N = 60\right)} \\
&= 1 - \frac{\mathbb{P}\left(N(15) = 0\right)\mathbb{P}\left(N(45) = 2\right)}{\mathbb{P}\left(N = 60\right)} \\
&= \frac{7}{16}
\end{aligned}$$

6. Global Balance Equations (it is easier to draw a state transition diagram first)

$$\frac{1}{2}\pi_s + \frac{1}{2}\pi_s = \frac{3}{8}\pi_r + \frac{3}{8}\pi_c$$
$$\frac{3}{8}\pi_c + \frac{3}{8}\pi_c = \frac{1}{2}\pi_s + \frac{3}{8}\pi_r$$
$$\frac{3}{8}\pi_r + \frac{3}{8}\pi_r = \frac{1}{2}\pi_s + \frac{3}{8}\pi_c$$
$$\pi_s + \pi_c + \pi_r = 1$$

From 2nd and 3rd equations, $\pi_c = \pi_r$. Hence

$$\pi_s = \frac{3}{11}$$
$$\pi_c = \pi_r = \frac{4}{11}$$