ELEN E6761: Communication Networks

Homework 0: Solutions

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1. $Y \sim \text{Poisson}(\mu), Z \sim \text{Geo}(p), Y \& Z : \text{indep.}$
   (a) $\mathbb{P}(Y = i | Y < Z)$ for $i \geq 0$:

$$\begin{aligned}
\mathbb{P}(Y = i | Y < Z) &= \frac{\mathbb{P}(Y < Z | Y = i)\,\mathbb{P}(Y = i)}{\mathbb{P}(Y < Z)} = \frac{\mathbb{P}(Z > i)\,\mathbb{P}(Y = i)}{\sum_{i=0}^{\infty} \mathbb{P}(Y < Z | Y = i)\,\mathbb{P}(Y = i)} \\
&= \frac{(1-p)^i \times \frac{e^{-\mu}\mu^i}{i!}}{\sum_{i=0}^{\infty}(1-p)^i \times \frac{e^{-\mu}\mu^i}{i!}} = \frac{((1-p)\mu)^i \times \frac{e^{-\mu}}{i!}}{e^{-\mu p}} \\
&= \frac{((1-p)\mu)^i \times e^{-(1-p)\mu}}{i!} \\
&\sim \text{Poisson}((1-p)\mu)
\end{aligned}$$

   (b) $\mathbb{E}[Y | Y < Z]$:

$$\begin{aligned}
\mathbb{E}[Y | Y < Z] &= \sum_{Y=0}^{\infty} y \times \mathbb{P}(Y = y | Y < Z) \\
&\overset{(a)}{=} \sum_{y=0}^{\infty} y \times \frac{((1-p)\mu)^y \times e^{-(1-p)\mu}}{y!} \\
&= (1-p)\mu
\end{aligned}$$

2. $X_1 \sim \text{Exp}(\lambda_1), X_2 \sim \text{Exp}(\lambda_2), Z = \min(X_1, X_2)$
   Let's find the CDF of Z.

$$\begin{aligned}
F_Z(z) &= \mathbb{P}(Z \leq z) = 1 - \mathbb{P}(Z > z) = 1 - \mathbb{P}(\min(X_1, X_2) > z) \\
&= 1 - \mathbb{P}(X_1 > z, X_2 > z) \overset{\text{indep.}}{=} 1 - \mathbb{P}(X_1 > z)\,\mathbb{P}(X_2 > z) \\
&= 1 - e^{-\lambda_1 z} \times e^{-\lambda_2 z} = 1 - e^{-(\lambda_1 + \lambda_2) z}
\end{aligned}$$

$$\begin{aligned}
f_Z(z) = F_Z'(z) &= \frac{dF_Z(z)}{dz} = (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2) z} \\
&\sim \text{Exp}(\lambda_1 + \lambda_2)
\end{aligned}$$

3. $X \& Y \sim \text{Exp}(\lambda), X \& Y : \text{indep.}, Z = |X - Y|$

$$\begin{aligned}
F_Z(z) &= \mathbb{P}(Z \leq z) = \mathbb{P}(|X - Y| \leq z) \\
&= \mathbb{P}(|X - Y| \leq z | X \geq Y)\,\mathbb{P}(X \geq Y) + \mathbb{P}(|X - Y| \leq z | X < Y)\,\mathbb{P}(X < Y) \\
&= \frac{1}{2} \times \mathbb{P}(X - Y \leq z | X \geq Y) + \frac{1}{2} \times \mathbb{P}(Y - X \leq z | X < Y)
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(X - Y \leq z | X \geq Y) &= \int_0^{\infty} \mathbb{P}(X - Y \leq z | X \geq Y, Y = y)\, f_Y(y) dy \\
&= \int_0^{\infty} \mathbb{P}(X - y \leq z | X \geq y)\, f_Y(y) dy \\
&\overset{*}{=} \int_0^{\infty} \mathbb{P}(X \leq z)\, f_Y(y) dy = \mathbb{P}(X \leq z) = 1 - e^{-\lambda z}
\end{aligned}$$

*: Exponential distribution is memoryless.

$\mathbb{P}(X - Y \leq z | X \geq Y)$ is equal to $\mathbb{P}(Y - X \leq z | X < Y)$, so:
$F_Z(z) = 1 - e^{-\lambda z} \longrightarrow f_Z(z) = \lambda e^{-\lambda z} \sim \text{Exp}(\lambda)$

4. $X \sim \text{Poisson}(\lambda_1), Y \sim \text{Poisson}(\lambda_2), X \& Y$: indep.
   $\mathbb{P}(X = k | X + Y = n)$:

$$\begin{aligned}
\mathbb{P}(X = k | X + Y = n) &= \frac{\mathbb{P}(X + Y = n | X = k)\,\mathbb{P}(X = k)}{\mathbb{P}(X + Y = n)} \\
&= \frac{\mathbb{P}(Y = n - k)\,\mathbb{P}(X = k)}{\sum_{s=0}^{n} \mathbb{P}(X + Y = n | X = s)\,\mathbb{P}(X = s)} \\
&= \frac{\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!} \times \frac{e^{-\lambda_1}\lambda_1^{k}}{(k)!}}{\sum_{s=0}^{n} \frac{1}{s!(n-s)!} e^{-\lambda_1}\lambda_1^{s} e^{-\lambda_2}\lambda_2^{n-s}} \\
&= \frac{\frac{\lambda_2^{n-k}}{(n-k)!} \times \frac{\lambda_1^{k}}{(k)!}}{\frac{1}{n!}\sum_{s=0}^{n} \frac{n!}{s!(n-s)!}\lambda_1^{s}\lambda_2^{n-s}} \\
&= \frac{n!}{k!(n-k)!} \frac{\lambda_1^{k}\lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^n} \\
&= \frac{n!}{k!(n-k)!} \frac{\lambda_1^{k}}{(\lambda_1 + \lambda_2)^k} \frac{\lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^{n-k}} \\
&\sim \text{Binomial}(n, \frac{\lambda_1}{\lambda_1 + \lambda_2})
\end{aligned}$$

5. Mean and Variance of random variables
   (a) $\phi(u) = \exp(-5u^2 + 2ju)$
   We know that $m_n = \frac{1}{j^n}\phi^{(n)}(u)|_{u=0}$ So:

$$\begin{aligned}
\mathbb{E}[X] =& \frac{1}{j}\frac{d\phi(ju)}{du}\Big|_{u=0} \\
=& \frac{1}{j}(-10u + 2j)exp(-5u^2 + 2ju)\Big|_{u=0} \\
=& 2. \\
\mathbb{E}\left[X^2\right] =& \frac{1}{j^2}\frac{d^2\phi(ju)}{du^2}\Big|_{u=0} \\
=& \Big(10exp(-5u^2 + 2ju) \\
& - (-10u + 2j)(-10u + 2j)exp(-5u^2 + 2ju)\Big)\Big|_{u=0} \\
=& 14. \\
\text{Var}(X) =& \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X] \\
=& 10.
\end{aligned}$$

In fact this is the characteristic function of a Gaussian $N(2, 10)$ random variable. (b) $\phi(u) = \frac{\exp(ju) - 1}{ju}$

$$\mathbb{E}[X] = \frac{1}{j}\frac{d\phi(ju)}{du}\bigg|_{u=0} = \frac{1}{j}\frac{-ue^{ju} - j(e^{ju}-1)}{(-u^2)}\bigg|_{u=0}$$

We should calculate limit:

$$\lim_{x\to 0}\frac{1}{j}\frac{-ue^{ju} - j(e^{ju}-1)}{(-u^2)} \overset{\text{l'Hopital}}{=} \lim_{x\to 0}\frac{1}{j}\frac{-e^{ju} - jue^{ju} + e^{ju}}{(-2u)} = \frac{1}{2}.$$

$$\mathbb{E}\left[X^2\right] = \frac{1}{j^2}\frac{d^2\phi(ju)}{du^2}\bigg|_{u=0} = \frac{-ju^2e^{ju} - 2ue^{ju} - 2je^{ju} + 2}{u^3}$$

We should calculate limit:

$$\lim_{x\to 0}\frac{-ju^2e^{ju} - 2ue^{ju} - 2je^{ju} + 2}{u^3} \overset{\text{l'Hopital}}{=} \frac{-j2ue^{ju} + u^2e^{ju} - 2e^{ju} - 2jue^{ju} + 2e^{ju}}{3u^2} = \frac{1}{3}$$

$$\text{Var}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X] = \frac{1}{12}$$

In fact, this is the characteristic function of a uniform $U[0,1]$ random variable.

6. X, Y, Z are random variables with finite expectations. $a, b \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.
(iv) $\mathbb{E}[X|Y] = \mathbb{E}[X]$:

$$\mathbb{E}[X|Y] = \sum_x x\mathbb{P}(X = x|Y) \overset{\text{indep.}}{=} \sum_x x\mathbb{P}(X = x) = \mathbb{E}[X]$$

(v) $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Z]]$:

$$\mathbb{E}[\mathbb{E}[X|Z]] = \mathbb{E}\left[\sum_x x\mathbb{P}(X = x|Z)\right] = \sum_z \Big(\sum_x x\mathbb{P}(X = x|Z = z)\Big)\mathbb{P}(Z = z)$$
$$= \sum_x x\Big(\sum_z \mathbb{P}(X = x|Z = z)\,\mathbb{P}(Z = z)\Big) = \sum_x x\mathbb{P}(X = x) = \mathbb{E}[X]$$

(vi) $\mathbb{E}[Xg(Y)|Y] = g(Y)\mathbb{E}[X|Y]$:
$\mathbb{E}[Xg(Y)|Y]$, itself is a random variable. And Y is a discrete random variable, so Assume $Y = y$:

$$\mathbb{E}[Xg(Y)|Y = y] = \mathbb{E}[Xg(y)|Y = y] \overset{*}{=} g(y)\mathbb{E}[X|Y = y] \text{ for all y}$$

*: Given $Y = y$, $g(Y)$ is determined. According to (ii) with a=g(y) and b=0, the result is obtained.

So $\mathbb{E}[Xg(Y)|Y] = g(Y)\mathbb{E}[X|Y]$

(vii) $\mathbb{E}[X|Y, g(Y)] = \mathbb{E}[X|Y]$:
$\mathbb{E}[X|Y, g(Y)]$, itself is a random variable, and Y is a discrete random variable. Then given

$Y = y$:

$$
\begin{aligned}
\mathbb{E}\left[X|Y=y, g(Y)=g(y)\right] &= \sum_x x\mathbb{P}\left(X=x|Y=y, g(Y)=g(y)\right) \\
&= \sum_x x\frac{\mathbb{P}\left(X=x\bigcap Y=y\bigcap g(Y)=g(y)\right)}{\mathbb{P}\left(Y=y, g(Y)=g(y)\right)} \\
&\overset{*}{=} \sum_x x\frac{\mathbb{P}\left(g(Y)=g(y)|X=x, Y=y\right)\mathbb{P}\left(X=x\bigcap Y=y\right)}{\mathbb{P}\left(Y=y\right)} \\
&\overset{**}{=} \sum_x x\frac{\mathbb{P}\left(X=x\bigcap Y=y\right)}{\mathbb{P}\left(Y=y\right)} \\
&= \sum_x x\mathbb{P}\left(X=x|Y=y\right) = \mathbb{E}\left[X|Y=y\right] \text{ for all y}
\end{aligned}
$$

*: $\mathbb{P}\left(Y=y, g(Y)=g(y)\right) = \mathbb{P}\left(Y=y\right)$
**: $\mathbb{P}\left(g(Y)=g(y)|X=x, Y=y\right) = \mathbb{P}\left(g(Y)=g(y)|Y=y\right) = 1$

So $\mathbb{E}\left[X|Y, g(Y)\right] = \mathbb{E}\left[X|Y\right]$