ELEN E6761: Communication Networks

Lecture 3: Poisson and Markov Processes Lecturer: Javad Ghaderi

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Definition 1 (Stochastic Process)**.** A stochastic process is sequence of random variables $(X_t : t \in T)$, where X_t takes on values in a set S. In many applications, the index set T is a set of times, and S is the set of possible states for the system. The index set T could be discrete-time (consecutive integers), or continuous-time (real numbers).

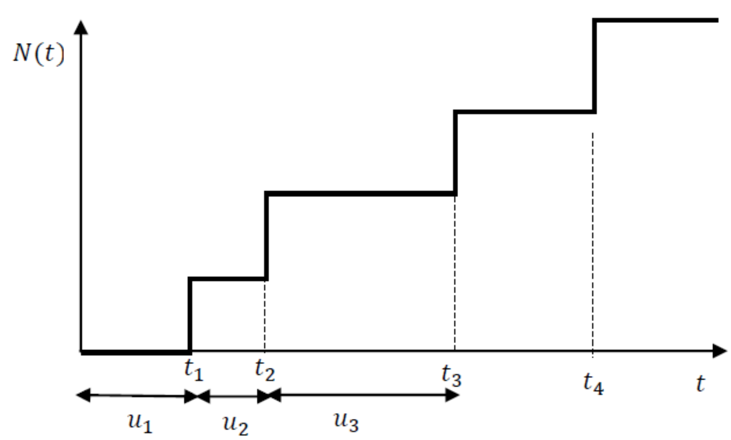
Any set of instances of $(X(t); t \in T)$ can be regarded as a path of a particle moving randomly in the state space S, its position at time t being $X(t)$. These paths are called sample paths of the stochastic process.

1 Counting Process and Poisson Process

Definition 2 (Counting Process)**.** A stochastic process is a counting process if its sample path is piecewise constant and increases by one at discrete instances in time (called count times or arrival times). Thus the process is described by a set of count (arrival) times $\{t_i\}$, or equivalently by the set of intercount (inter-arrival) times $u_1 = t_1, \cdots, u_i = t_i - t_{i-1}, \cdots$. Hence, the number of arrivals before time t, $N(t)$, is

$$N(t) = \sum_i \mathbb{1}(t_i \leq t),$$

where $\mathbb{1}(\cdot)$ is the indicator function.



Definition 3 (Poisson Process)**.** Consider a counting process $N(t)$ with $N(0) = 0$. If intercount times u_i, $i = 1, 2, \cdots$, are iid exponentially distributed with parameter λ, then $N(t)$ is called a Poisson process with rate λ.

It follows that this definition of Poisson process is equivalent to the following two definitions.

Definition 4 (Poisson Process)**.** A counting process $N(t)$ is called a Poisson process with rate λ if
(i)$N(0) = 0$,
(ii)$N(t+s) - N(t)$ is independent of $N(t)$ (independent increments property),
(ii)$N(t+s) - N(s)$ has a Poisson distribution with parameter λs, i.e.,

$$\mathbb{P}\left\{N(t+s) - N(s) = k\right\} = \frac{(\lambda s)^k e^{-\lambda s}}{(\lambda s)!}, k = 0, 1, 2, \cdots$$

Another equivalent definition is the following.

Definition 5 (Poisson Process)**.** A counting process $N(t)$ is called a Poisson process with rate λ if

1. $N(0) = 0$.
2. $\mathbb{P}\left\{N(t+\delta) - N(t) = 1\right\} = \lambda\delta + \mathrm{o}(\delta)$,
3. $\mathbb{P}\left\{N(t+\delta) - N(t) > 1\right\} = \mathrm{o}(\delta)$.

Note that $\mathrm{o}(\cdot)$ means that $\frac{\mathrm{o}(\delta)}{\delta} \to 0$ as $\delta \to 0$ (for example $\delta^2 = \mathrm{o}(\delta)$). Hence in any small enough interval of length δ, there is either a count with probability $\lambda\delta$, or no counts with probability $1 - \lambda\delta$ (only these two events have non-negligible probability).

1.1 Properties of Poisson Process

Memoryless property

Assume we have waited for some time 's' after t_i, what is the probability that we have to wait for an additional 't' time units before seeing the $(i+1)$-th count?

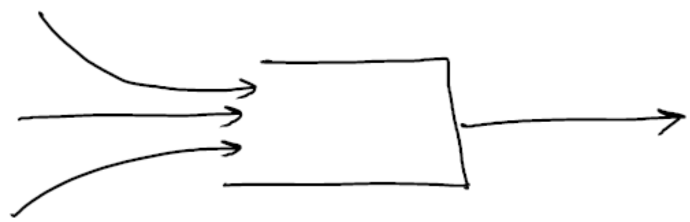
$$\mathbb{P}\left\{t_{i+1} > t_i + s + t | t_{i+1} > t_i + s\right\} = \frac{\mathbb{P}\left\{(t_{i+1} > t_i + s + t) \cap (t_{i+1} > t_i + s)\right\}}{\mathbb{P}\left\{t_{i+1} > t_i + s\right\}} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t}$$

i.e., the waiting time is still exponential independent of s.

Merging

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates λ_1 and λ_2. Then the process $N_1(t) + N_2(t)$ is Poisson with rate $\lambda_1 + \lambda_2$.

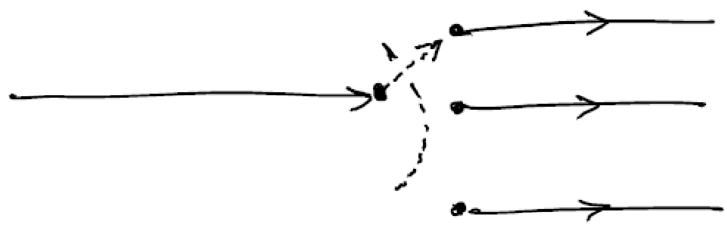
Clearly this can be extended to more than two processes.



Splitting

Suppose $N(t)$ is a Poisson process with rate λ. Create two new process $N_1(t)$ and $N_2(t)$ by assigning each count event to the first process with probability p and to the second one with probability $1-p$. Then $N_1(t)$ and $N_2(t)$ are independent Poisson processed with rates λp and $\lambda(1-p)$ respectively.

Clearly this can be extended to slitting into more than two processes.



2 Markov processes

Markov Processes naturally arise in the modeling of many systems where there is a notion of state for the system at each time. The state at time t contains all the relevant information about the system up to and including time t that is relevant to the future of the system. For example, the state of an aircraft at time t could consist of the position, velocity, and remaining fuel at time t. Think of t as the present time. Given the state at time t, the future part of the aircraft trajectory is determined, independently of the history up to time t, i.e, it does not matter how the aircraft has reached the current state at time t.

Definition 6 (Markov Process)**.** A process $X(t)$ is a Markov process if it has the memoryless property: Given the value of $X(t)$ at some time $t \in T$, the future path $X(s)$ for $s > t$ does not depend on knowledge of the past history $X(u)$ for $u < t$, i.e. for $t_1 < \cdots < t_n < t_{n+1}$,

$$\mathbb{P}\left\{X(t_{n+1}) = x_{n+1} | X(t_n) = x_n; \cdots ; X(t_1) = x_1\right\} = \mathbb{P}\left\{X(t_{n+1}) = x_{n+1} | X(t_n) = x_n\right\}$$

The Markov processes that we will be considering in the course will have the following properties.

Definition 7 (Irreducibility)**.** $X(t)$ is irreducible if all states in S can be reached from all other states, by following the transitions of the process. If we draw a directed graph of the state space with a node for each state and an arc for each event, or transition, then for any pair of nodes there is a path connecting them, i.e. the graph is strongly connected.

Definition 8 (Time-homogeneity)**.** $X(t)$ is time homogeneous if behavior of the system does not depend on when it is observed. In particular, the transition probabilities between states are independent of the time at which the transitions occur. Thus, for all s and u,

$$\mathbb{P}\left\{X(s+t_1)=y|X(s)=x\right\}=\mathbb{P}\left\{X(u+t_1)=y|X(u)=x\right\}.$$

In this course, our primary objective with respect to a Markovian models will be to calculate the probability distribution of the random variable $X(t)$ over the state space S, as the time goes to infinity. The long run behavior of the system usually approaches a regular pattern called "the steady-state probability distribution". From this probability distribution we will derive performance measures based on subsets of states where some condition holds.